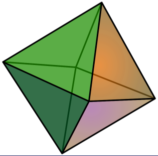
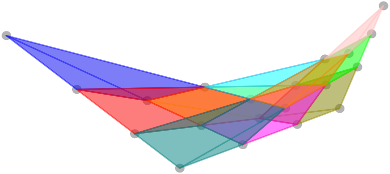
Eulerian numbers and the $m = 2$ amplituhedron: signs and triangulations from the hypersimplex

Lauren K. Williams, Harvard

Based on:

- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Tomasz Lukowski and Matteo Parisi, arXiv:2002.06164
- Work in preparation with Matteo Parisi and Melissa Sherman-Bennett
- (Previous works with Steven Karp and Karp-Zhang.)





Program

- Geometry of Grassmannian and matroid stratification. GGMS '87. Hypersimplex and matroid polytopes.
- Add *positivity* to the previous picture. Postnikov '06. Positroid stratification of $(Gr_{kn})_{\geq 0}$, positroid polytopes, plabic graphs.
- Simultaneous generalization of $(Gr_{k,n})_{\geq 0}$ and polygons: amplituhedron. Arkani-Hamed and Trnka '13.
- Thesis: ideas and results about the hypersimplex and positroid polytopes have parallels for the amplituhedron.
- Twistor coordinates and sign flips. Amplituhedron sign stratification and Eulerian numbers.
- The hypersimplex and the $m = 2$ amplituhedron: T-duality and positroid triangulations.

The Grassmannian and the matroid stratification

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The *matroid* associated to $A \in Gr_{k,n}$ is $\mathcal{M}(A) := \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0.\}$

Gelfand-Goreksy-MacPherson-Serganova introduced the *matroid stratification* of $Gr_{k,n}$: given $\mathcal{M} \subset \binom{[n]}{k}$, the *matroid stratum* $S_{\mathcal{M}}$ is

$$S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid p_I(A) \neq 0 \text{ iff } I \in \mathcal{M}\}.$$

Have the matroid stratification

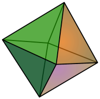
$$Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}.$$

The Grassmannian and the moment map

Recall: given $\mathcal{M} \subset \binom{[n]}{k}$, the *matroid stratum* S_M is $S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid p_I(A) \neq 0 \text{ iff } I \in \mathcal{M}\}$.

Let $\{e_1, \dots, e_n\}$ be basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is $\quad \mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n$.



Let $\Delta_{k,n} := \mathrm{Conv}\{e_I \ : \ |I| = k\} \subset \mathbb{R}^n$ be the *hypersimplex.*

GGMS: the moment map image $\overline{\mu(S_{\mathcal{M}})}$ of the matroid stratum $S_{\mathcal{M}}$ is the *matroid polytope* $\Gamma_{\mathcal{M}} := \mathrm{Conv}\{e_I \mid I \in \mathcal{M}\}$. And $\mu(Gr_{k,n}) = \Delta_{k,n}$.

Remark: (normalized) volume of $\Delta_{k,n}$ is the *Eulerian number*, the numbers of permutations on S_{n-1} with $k-1$ descents.

What if we add the adjective "positive" to the whole story?

Background: Lusztig's theory of total positivity for G/P 1994, Rietsch 1997, Postnikov's 2006 preprint on the *totally non-negative* (TNN) or "positive" Grassmannian.

Let $(Gr_{k,n})_{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

Inspired by matroid stratification, one can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$(Gr_{k,n})_{\geq 0} = \sqcup S_{\mathcal{M}}^{tnn}.$$

What if we add the adjective "positive" to the whole story?

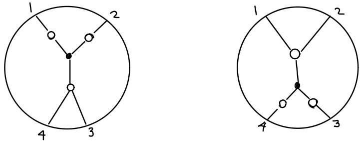
- Recall: *matroid* assoc to $A \in Gr_{k,n}$ is $\mathcal{M}(A) := \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0.\}$
- And the *matroid polytope* is $\Gamma_{\mathcal{M}} = \text{Conv}\{e_I \mid I \in \mathcal{M}.\}$
- If $A \in (Gr_{k,n})_{\geq 0}$, call $\mathcal{M}(A)$ a *positroid* and $\Gamma_{\mathcal{M}}$ a *positroid polytope*.
- Can restrict moment map from $Gr_{k,n}$ to $(Gr_{k,n})_{\geq 0}$: each positroid polytope is moment map image of positroid cell. (Tsukerman-W.)

Theorem (Postnikov)

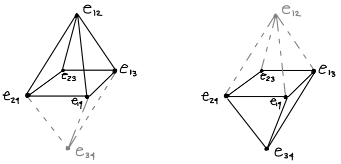
The positroid cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with *decorated permutations* π on $[n]$ with k antiexcedances. Also in bijection with equivalence classes of *planar bicolored (plabic) graphs*, or *on-shell graphs*.

How to read off a positroid (polytope) from a plabic graph

- Positroid cells $\leftrightarrow$ *plabic graphs*, planar bicolored graphs embedded in disk with boundary vertices labeled $1, 2, \dots, n$ and internal vertices colored black or white.



- WLOG we assume graph G is bipartite and that every boundary vertex is incident to a white vertex.
- Let $\mathcal{M}(G) := \{\partial(P) \mid P \text{ is a perfect matching of } G\}$.
- $\mathcal{M}(G)$ a positroid, and all positroids obtained this way (Postnikov).



Background and Motivation for the amplituhedron

- Introduced by Arkani-Hamed and Trnka in 2013.
- The amplituhedron is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$:

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k + m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}(Z)$ depends on Z but its combin. properties appear not to.
- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" is supposed to compute scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory;
  the *BCFW recurrence* for scattering amplitudes can be reformulated as giving a triangulation of the $m = 4$ amplituhedron.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)
- $m = 4$: case of main physical interest.
- $m = 2$: toy model for $m = 4$ – and connected to hypersimplex! ...

Twistor coordinates for the amplituhedron

Twistor coordinates

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Denote rows of Z by $Z_1, \ldots, Z_n$. Given a matrix Y with rows $y_1, \ldots, y_k$ representing element of $Gr_{k,k+m}$, and $1 \leq i_1 < \cdots < i_m \leq n$, let

$$\langle YZ_{i_1} \ldots Z_{i_m} \rangle = \langle y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m} \rangle$$

(determinant of a $(k+m) \times (k+m)$ matrix). Call it a *twistor coordinate*.

Lemma

An element $Y \in Gr_{k,k+m}$ is uniquely determined by its twistor coordinates.

Recall that $\mathcal{A}_{n,k,m}(Z)$ is the image of $(Gr_{k,n})_{\geq 0}$ in $Gr_{k,k+m}$.
Arkani-Hamed–Thomas–Trnka conjectured that the amplituhedron can be described directly in $Gr_{k,k+m}$ using twistor coordinates.
True for $m = 1$ (Karp-W.)

Theorem: sign flip description of $\mathcal{A}_{n,k,2}$

Given a sequence $(r_1, \dots, r_n)$ of real numbers, define $\mathrm{var}(r_1, \dots, r_n)$ to be the number of sign flips as we read left to right (ignoring 0's).
E.g. $\mathrm{var}(2, -1, 0, -2, 3, -1) = 3$.

Theorem (Parisi–Sherman-Bennett-W.)

Fix $k < n$ and $m = 2$, and $Z \in \mathrm{Mat}^{+}_{n,k+2}$. Let

$$\begin{aligned}\mathcal{F}^{\circ}_{n,k,2}(Z) := \{Y \in Gr_{k,k+2} \mid \langle YZ_iZ_{i+1}\rangle > 0 \text{ for } 1 \le i \le n-1, \\ (-1)^{k-1}\langle YZ_nZ_1\rangle > 0, \\ \text{and } \mathrm{var}(\langle YZ_1Z_2\rangle, \langle YZ_1Z_3\rangle, \dots \langle YZ_1Z_n\rangle) = k.\}\end{aligned}$$

Then $\mathcal{A}_{n,k,2}(Z) = \overline{\mathcal{F}^{\circ}_{n,k,2}(Z)}$.

Note:Arkani-Hamed–Thomas–Trnka conjectured this was true and sketched an argument that $\mathcal{A}_{n,k,2}(Z) \subseteq \overline{\mathcal{F}^{\circ}_{n,k,2}(Z)}$; Karp–W. gave an independent proof of this direction. We prove the converse.

Amplituhedron stratification and Eulerian numbers

- Def: the **amplituhedron sign stratification** is the partition of the amplituhedron into strata based on signs of twistor coordinates.[1]
- $m = 2$: let $\sigma = (\sigma_{ij}) \in \{0, +, -\}^{\binom{n}{2}}$ be a sign vector. Set $\mathcal{A}^{\sigma}_{n,k,2}(Z) := \{Y \in \mathcal{A}_{n,k,2}(Z) : \text{sign}\langle Y Z_i Z_j\rangle = \sigma_{ij}\}$.
- Call $\mathcal{A}^{\sigma}_{n,k,2}(Z)$ an *amplituhedron sign stratum* of $\mathcal{A}_{n,k,2}(Z)$. If $\sigma \in \{+, -\}^{\binom{n}{2}}$, call $\mathcal{A}^{\sigma}_{n,k,2}(Z)$ an *amplituhedron (sign) chamber.*
- Note: when $m > 1$, many amplituhedron strata are empty. Whether or not $\mathcal{A}^{\sigma}_{n,k,2}(Z)$ is empty depends on Z. Say σ is **realizable** for $\mathcal{A}_{n,k,2}$ if $\mathcal{A}^{\sigma}_{n,k,2}(Z)$ is nonempty for some Z.

Proposition (Parisi–Sherman-Bennett-W.)

All $\binom{n}{2}$ amplituhedron sign chambers in $\mathcal{A}_{n,k,2}$ are empty *except* for a collection indexed by the permutations $\{w \in S_{n-1} \mid w \text{ has } k \text{ descents}\}$.
These permutations are counted by the *Eulerian numbers*.
Note: the volume of the hypersimplex is the Eulerian number!

[1]studied for $m = 1$ by Karp-W.

w-chambers in $\mathcal{A}_{n,k,2}$ and w-simplices in $\Delta_{k+1,n}$

- Recap: given $\sigma = (\sigma_{ij}) \in \{\pm\}^{\binom{n}{2}}$ a sign vector, we define the *amplituhedron sign chamber* $\mathcal{A}^{\sigma}_{n,k,2}(Z) := \{Y \in \mathcal{A}_{n,k,2}(Z) : \text{sign}\langle YZ_iZ_j\rangle = \sigma_{ij}\}$.
  Most are empty except ...
- Let $w = (w_1, w_2, \dots, w_n = n) \in S_n$ with $k+1$ *cyclic descents*. Let $I_r := \{j \in [n] : j$ a cyc descent of the rotation of w ending at $r-1\}$.
- Define a corresponding amplituhedron w-**chamber** by $\hat{\Delta}_w(Z) := \{Y \in Gr_{k,k+2} \mid \text{sign}\langle YZ_aZ_j\rangle = (-1)^{|I_a \cap [a+1,j]|}\ \forall a \neq j\}$.[2]
- **Theorem (P–SB–W)**: We have $\mathcal{A}_{n,k,2}(Z) = \cup_w \hat{\Delta}_w(Z)$.
- Given w as above, we also define the hypersimplex w-**simplex** as $\Delta_w := \text{Conv}\{e_{I_1}, \dots, e_{I_n}\} \subset \Delta_{k+1,n}$.
- **Theorem (Stanley '77)**: We have $\Delta_{k+1,n} = \cup_w \Delta_w$. [3]
- Not a coincidence ...

[2] RHS gets multiplied by -1 if $a > j$ and k even

[3] see also Sturmfels '96 and Lam-Postnikov '07

Theorem: characterization of gen. triangles of $\mathcal{A}_{n,k,2}$

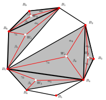
The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \mathrm{Mat}^+_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

"Generalized triangles" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- If S_π a positroid cell of $(Gr_{k,n})_{\geq 0}$ such that $\dim S_\pi = km$ and $\tilde{Z}$ is injective on S_π, call $Z_\pi := \overline{\tilde{Z}(S_\pi)}$ a **generalized triangle** for $\mathcal{A}_{n,k,m}$.

- A (k, n)-unpunctured plabic tiling in n-gon is collection of noncrossing black polygons which can be triangulated into k black triangles.

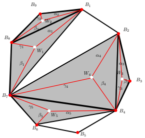


Theorem: characterization of gen. triangles of $\mathcal{A}_{n,k,2}$

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- If S_π a positroid cell of $(Gr_{k,n})_{\geq 0}$ such that $\dim S_\pi = km$ and $\tilde{Z}$ is injective on S_π, call $Z_\pi := \overline{\tilde{Z}(S_\pi)}$ a **generalized triangle** for $\mathcal{A}_{n,k,m}$.

Theorem (P–SB–W): characterization of GT's of $\mathcal{A}_{n,k,2}$

Let $\mathcal{T}$ be a (k,n)-unpunctured plabic tiling in n-gon. (any triangulation of it uses k black triangles). This gives a $2k$-dimensional positroid cell $S_\mathcal{T}$ on which $\tilde{Z}$ is injective, and all generalized triangles arise in this way.



This proved a conjecture of Lukowski-Parisi-Spradlin-Volovich.

One can also define GT's for the moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$; they are in bijection with GT's for $\mathcal{A}_{n,k,2}$! (Lukowski–Parisi–W).

Positroid triangulations of the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^+_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Positroid "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- We say that $Z_\pi := \overline{\tilde{Z}(S_\pi)}$ is a **generalized triangle** if $\dim S_\pi = km$ and $\tilde{Z}$ is injective on S_π.
- Problem: Find collection of generalized triangles $\{Z_\pi\}$ whose interiors are pairwise disjoint, and whose union equals $\mathcal{A}_{n,k,m}(Z)$.
  Call this a **(positroid) triangulation** of $\mathcal{A}_{n,k,m}(Z)$.

Positroid triangulations of the hypersimplex

The hypersimplex $\Delta_{k,n} = \mathrm{Conv}\{e_I \mid I \in \binom{[n]}{k}\}$

Fix k, n. Have the **moment map** $\mu : (Gr_{k,n})_{\geq 0} \to \mathbb{R}^n$:

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The hypersimplex $\Delta_{k,n} = \mu((Gr_{k,n})_{\geq 0}) \subset \mathbb{R}^n$.

Positroid "triangulations" of the hypersimplex

- Have $\dim \Delta_{k,n} = n-1 \leq \dim(Gr_{k,n})_{\geq 0}$, so μ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- We say that $\Gamma_\pi := \overline{\mu(S_\pi)}$ is a **generalized triangle** if $\dim S_\pi = n-1$ and μ is injective on S_π.
- Problem: Find collection of generalized triangles $\{\Gamma_\pi\}$ whose interiors are pairwise disjoint, and whose union equals $\Delta_{k+1,n}$.
  Call this a **(positroid) triangulation** of $\Delta_{k+1,n}$.

(Positroid) triangulations of $\mathcal{A}_{n,1,2}$

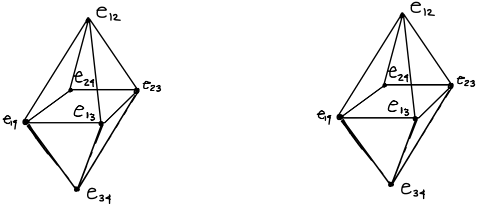
- Recall that $\mathcal{A}_{n,1,2}$ is a polygon (n-gon) in projective space $\mathbb{P}^2$.

- Positroid triangulations of $\mathcal{A}_{n,1,2}$ are ordinary triangulations of the n-gon
- Each triangulation consists of $n-2$ triangles, each of dimension 2
- The total number of triangulations of $\mathcal{A}_{n,1,2}$ is the *Catalan number* $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.
- When $n=2$, have two triangulations of $\mathcal{A}_{n,1,2}$ (quadrilateral).

(Positroid) triangulations of $\Delta_{2,n}$

- Each positroid triangulation consists of $n-2$ positroid cells ("triangles"), each of full dimension $n-1$;
- The total number of positroid triangulations of $\Delta_{2,n}$ is the *Catalan number* $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ (Speyer-W.)

Example: $\mu : (Gr_{2,4})_{\geq 0} \to \Delta_{2,4} \subset \mathbb{R}^4$



Comparison with $\mathcal{A}_{n,1,2}$.

T-duality and a weird conjecture

Conjecture (Lukowski-Parisi-W.)

Positroid triangulations of the amplituhedron $\mathcal{A}_{n,k,2}$ are in bijection with positroid triangulations of the hypersimplex $\Delta_{k+1,n}$. Bijection TBD ...

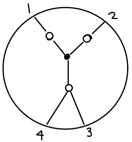
- Triangulations of $\Delta_{k+1,n}$ come from $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$, while triangs of $\mathcal{A}_{n,k,2}$ come from $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

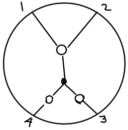
Recall that cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ decorated permutations π on $[n]$ with k antiexcedances.

Indexing of positroid cells by permutations

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

- A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
- Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
- One can read off description of cell S_π from π.
- Given (reduced) plabic graph representing positroid cell, can read off permutation π by following "rules of road": right at black, left at white.





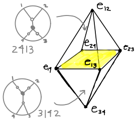
T-duality map on positroid cells

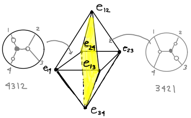
T-duality: given loopless dec perm $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

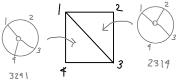
- Lukowski–Parisi–W.: The T-duality map $S_\pi \leftrightarrow S_{\hat{\pi}}$ is a bijection: loopless cells of $(Gr_{k+1,n})_{\geq 0} \leftrightarrow$ coloopless cells of $(Gr_{k,n})_{\geq 0}$,
- Parisi–Sherman-Bennett–W.: Moreover it is a poset isomorphism.

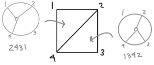
Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.





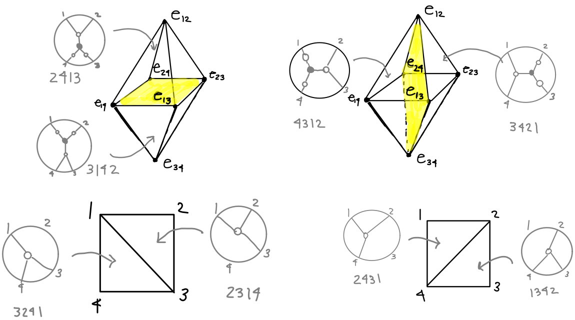




Given loopless decorated permutation $\pi = (a_1, \dots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \dots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

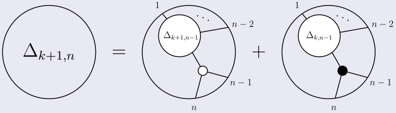
A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.



Conjecture true for infinitely many triangulations

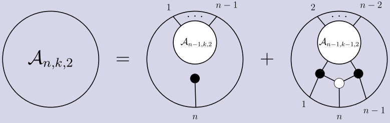
Theorem (Lukowski–Parisi–W.)

The following recursion constructs triangulations of $\Delta_{k+1,n}$ in terms of triangulations of $\Delta_{k+1,n-1}$ and $\Delta_{k,n-1}$:



Theorem (Bao-He)

The following recursion constructs triangulations of $\mathcal{A}_{n,k,2}$ in terms of triangulations of $\mathcal{A}_{n-1,k,2}$ and $\mathcal{A}_{n-1,k-1,2}$:



Theorem (L-P-W): These recursions are in bijection via T-duality.

One direction of conjecture always true

Theorem (Parisi–Sherman-Bennett–W.)

Suppose that a collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation $\{\Gamma_\pi\}$ of $\Delta_{k+1,n}$. Then the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation $\{Z_{\hat{\pi}}\}$ of $\mathcal{A}_{n,k,2}$.

- Let $w = (w_1, w_2, \ldots, w_n = n) \in S_n$ with $k+1$ *cyclic descents*. Let $I_r = \{j \in [n] : j$ a cyclic descent of the rotation of w ending at $r-1\}$.
  - The hypersimplex *w-simplex* is $\Delta_w := \text{Conv}\{e_{I_1}, \ldots, e_{I_n}\} \subset \Delta_{k+1,n}$.
  - The corresponding amplituhedron stratum is $\hat{\Delta}_w(Z) := \{Y \in Gr_{k,k+2} \mid \text{sign}\langle Y Z_a Z_j \rangle = (-1)^{|I_a \cap [a+1,j]|} \; \forall a \neq j\}$.[4]
- Claim: $\Delta_w \subset \Gamma_\pi \subset \Delta_{k+1,n}$ iff $\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}} \subset \mathcal{A}_{n,k,2}(Z)$.
- Only proves conjecture in one direction because depending on Z, $\hat{\Delta}_w(Z)$ could be empty!

[4] RHS gets multiplied by -1 if $a > j$ and k even

Table: Correspondences via T-duality (Parisi–Sherman-Bennett–W.)

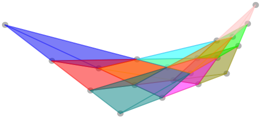
The Hypersimplex $\Delta_{k+1,n}$	VS	The Amplituhedron $\mathcal{A}_{n,k,2}$
	GENERALISED TRIANGLES	
$\Gamma_{G(\mathcal{T})}$ (Positroid Polytope)	unpunctured plabic tiling $\mathcal{T}$	(Grasstope) $Z_{\hat{G}(\mathcal{T})}$
$x_{[h,j-1]} \geq \text{area}_{\mathcal{T}}(h \to j)$	compatible arc $h \to j$	$\text{sign}\langle Yhj \rangle = (-1)^{\text{area}_{\mathcal{T}}(h\to j)}$
$x_{[h,j-1]} = \text{area}_{\mathcal{T}}(h \to j)$	facet defining arc $h \to j$	$\langle Yhj \rangle = 0$
	w-SIMPLICES and w-CHAMBERS	
simplex $\Delta_w \subset \Delta_{k+1,n}$	$w \in S_n$: $w(n) = n$, $\#\text{cdes}(w) = k+1$	chamber $\hat{\Delta}_w \subset \mathcal{A}_{n,k,2}$ such that:
$\Delta_w := \text{Conv}\{e_{I_1}, \dots, e_{I_n}\}$	$I_a = I_a(w) := \text{cdes}(\text{rot}(w, a-1))$	$\text{Flip}(\langle Ya\hat{1}\rangle, \langle Ya\hat{2}\rangle, \dots, \langle Yan\rangle) = I_a \setminus \{a\}$
$\Delta_{k+1,n} = \bigcup_w \Delta_w$		$\mathcal{A}_{n,k,2}(Z) = \bigcup_w \hat{\Delta}_w(Z)$
$\Delta_w \subset \Gamma_\pi$	$\Leftrightarrow$	$\hat{\Delta}_w \subset Z_{\hat{\pi}}$
	TRIANGULATIONS	
$\{\Gamma_\pi\}$ triangulates $\Delta_{k+1,n}$	$\Rightarrow$	$\{Z_{\hat{\pi}}\}$ triangulates $\mathcal{A}_{n,k,2}$
Eulerian triangulation $\{\Gamma_\pi\}$	positions of descents/sign flips	sign flip (BCFW) triangulation $\{Z_{\hat{\pi}}\}$

Summary

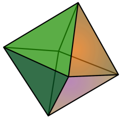
- The hypersimplex $\Delta_{k+1,n} \subset \mathbb{R}^n$ is the image of $(Gr_{k+1,n})_{\geq 0}$ under the moment map μ. It is a polytope of dimension $n-1$.
- The amplituhedron $\mathcal{A}_{n,k,2}(Z) \subset Gr_{k,k+2}$ is the image of $(Gr_{k,n})_{\geq 0}$ under the amplituhedron map $\tilde{Z}$. It is not a polytope and it has dimension $2k$.
- Nevertheless, $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are closely related via T-duality $\pi \mapsto \hat{\pi}$:
  - They both have natural decompositions into simplices/chambers counted by the Eulerian numbers.
  - The moment map μ is injective on an $(n-1)$-dimensional positroid cell S_π iff $\tilde{Z}$ is injective on the $2k$-dimensional positroid cell $S_{\hat{\pi}}$.
  - $\Delta_{k+1,n}$ and its generalized triangles can be described by inequalities; $\mathcal{A}_{n,k,2}$ and its generalized triangles have a parallel description using signs of twistor coordinates.
  - Every positroid triangulation of $\Delta_{k+1,n}$ gives rise to a positroid triangulation of $\mathcal{A}_{n,k,2}(Z)$ under T-duality.

Thank you for listening!

I. Amplituhedron '13



II. Hypersimplex and moment map '87



- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Lukowski and Parisi, arXiv:2002.06164
- "Eulerian numbers and the $m = 2$ amplituhedron: sign flips and triangulations," with Parisi and Sherman-Bennett, in preparation.