The amplituhedron for biadjoint scalar ϕ^3 theory: the associahedron

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Scattering amplitudes

The traditional approach to scattering amplitudes proceeds by writing down a lot of Feynman diagrams and summing them.

The geometric/amplituhedral approach to the first step is to describe the integrand as the “canonical differential form” of a geometric object (the “amplituhedron”).

You will have noticed that Feynman diagrams are completely hidden in the story as discussed by Steven and Lauren. An appealing feature of the ϕ^3 story that I am going to try to tell is that it is simple enough that we will actually be able to see both the Feynman diagrams and the amplituhedron at once.

Outline for this talk

I am going to describe for you the object that plays the rôle of the amplituhedron for biadjoint scalar ϕ^3. It turns out that this object is much simpler than the amplituhedron which Steven and Lauren have been talking about: it is a polytope (the convex hull of finitely many points in $\mathbb{R}^n$).

I am therefore going to spend a fair bit of time explaining how to think about the canonical differential form associated to a polytope.

I am also going to explain exactly what form we want to produce, viewed as a sum over Feynman diagrams.

I will then describe the ϕ^3 amplituhedron. It turns out to be a familiar polytope: the associahedron, originally developed by Jim Stasheff in the 60s. I will sketch how its canonical form agrees with the sum over Feynman diagrams.

Time permitting, I will say a little bit about how to think about what I have said from a more cluster-y perspective, which also gives some insights into how to produce amplituhedra for 1-loop and higher corrections.

References

I should say up front that very little of what I am going to be discussing is my own work.

The main source I will be drawing on is the paper by Arkani-Hamed, Bai, He, and Yan which constructed the amplituhedron for ϕ^3 theory: arXiv:1711.09102.

The reference for positive geometries more generally is the paper by Arkani-Hamed, Bai, and Lam: arXiv:1703.04541.

Two references for the more cluster-y stuff at the end are:

- Bazier-Matte, Chapelier, Douville, Mousavand, T., Yıldırım, arXiv:1808.09986 and
- Arkani-Hamed, He, Salvatori, T., arXiv:1912.12948

Canonical forms for positive geometries

The definition of what exactly a positive geometry is, and what its canonical form is, will be discussed by Thomas.

In the general setting, we have a d-dimensional complex algebraic variety X, and a positive part $X_{\geq 0}$, which is a semi-algebraic set inside the real points of X. When $(X, X_{\geq 0})$ is a positive geometry, there is a uniquely determined d-form, $\Omega(X, X_{\geq 0})$, on X, called its canonical form.

The amplituhedra inside Grassmannians are (conjecturally) examples of positive geometries; their canonical forms come from pushing forward the differential forms on BCFW cells.

A full-dimensional polyhedron P in $\mathbb{R}^d$ is also an example of a positive geometry. In this case $X = \mathbb{CP}^d$ and $X_{\geq 0} = P$; we drop X from the notation, writing $\Omega(P)$.

Canonical form of a polyhedron

Let P be a full-dimensional polyhedron. $\Omega(P)$ is defined inductively. The canonical form of a point is defined to be 1. From there $\Omega(P)$ is defined by

- For each facet F (codimension 1 face) of P, suppose that F is cut out by the hyperplane $f(x) = 0$ and P is on the positive side of F. We require that the residue of $\Omega(P)$ along $f(x) = 0$ is $\Omega(F)$. That is to say:

  $$\Omega(P) = \Omega(F) \wedge \frac{df}{f} + \cdots$$

  where the $\cdots$ denotes terms which remain smooth as f goes to zero.
- We further require that $\Omega(P)$ has no other singularities.

In fact this is the same definition as in the general setting, except that in general, the equation cutting out the boundary won't necessarily be an affine-linear equation.

Examples

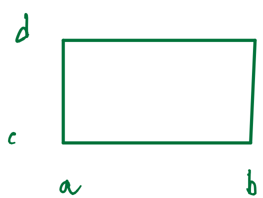
The canonical form of the interval $[a, b]$ is

$$\Omega([a,b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{(b-a)}{(b-x)(x-a)}dx$$

We can check that its residue at a and b is 1, and it has singularities only at a and b.

The canonical form of a product of two intervals is

$$\Omega([a,b] \times [c,d]) = \frac{(b-a)}{(b-x)(x-a)}dx \wedge \frac{(c-d)}{(d-y)(y-c)}dy$$



Canonical form of a simple polytope as a sum

A polytope P is called simple if the number of facets meeting at each vertex is d, the dimension of the polytope. Suppose that P is simple.

Let us write $V(P)$ for the vertices of P. For each vertex $v \in V(P)$, let us number the facets meeting at v as $F_1^{(v)}, \ldots, F_d^{(v)}$.

For F a facet of P, write f_F for a linear form such that F lies in the hyperplane $f_F = 0$.

$$\Omega(P) = \sum_{v \in V} \operatorname{sign}(F_1^{(v)}, \ldots, F_d^{(v)}) \bigwedge_{i=1}^{d} \frac{df_{F_i^{(v)}}}{f_{F_i^{(v)}}}$$

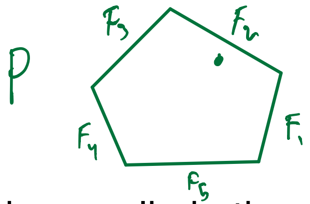
Here the sign comes from the orientation of the outward facet normals.

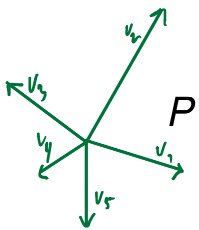
That the formula satisfies the definition follows by induction: clearly the only poles are along the faces $f_F = 0$, and along such a face, the contributions which aren't smooth add up to give (by induction) $\frac{df_F}{f_F} \wedge \Omega(F)$.

Canonical form of a polytope as a volume

There is another interpretation of the canonical form of a polytope.

If P is a polytope containing the origin in a real vector space V, there is a well-defined irredundant expression for P as an intersection of half-spaces:





$$P = \bigcap_{i=1}^{n} \{x \mid \langle x, v_i \rangle \geq -1\}$$

where v_i lie in the dual space V^*, and $\langle \cdot, \cdot \rangle$ is the pairing between V and V^*.

Define the dual polytope to P, denoted $P^\vee$, to be the convex hull of v_i.

Now $\Omega(P)$ can be defined as follows:

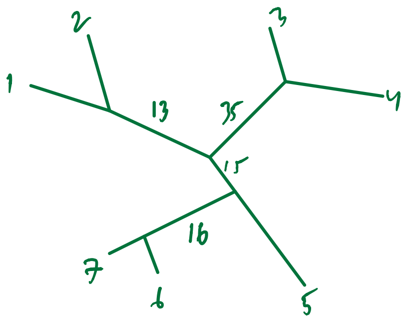
$$\Omega(P)(y) = d!\,\mathrm{Vol}((P-y)^\vee)dy_1 \wedge \cdots \wedge dy_d$$

(The RHS is defined for y inside P as I set it up but it's rational so it gives you something well-defined on V.)

The Feynman diagrams

Now we are going to switch gears and look at the physics input from the particular QFT we are considering.

We are going to consider $\mathcal{D}_n$, the planar, trivalent trees with n external nodes numbered cyclically clockwise 1 to n.



To each internal edge, we associate the pair $1 \leq i < j \leq n$ where the vertices on the side not including n are $i, \ldots, j-1$.

The external nodes each have an associated momentum p_i. We arrange things so that $\sum p_i = 0$. We associate to the internal edge e_{ij} the quantity:

$$X_{ij} = (p_i + \cdots + p_{j-1})^2.$$

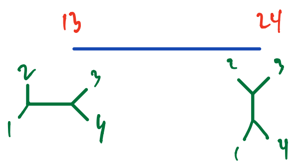
The Feynman diagram sum

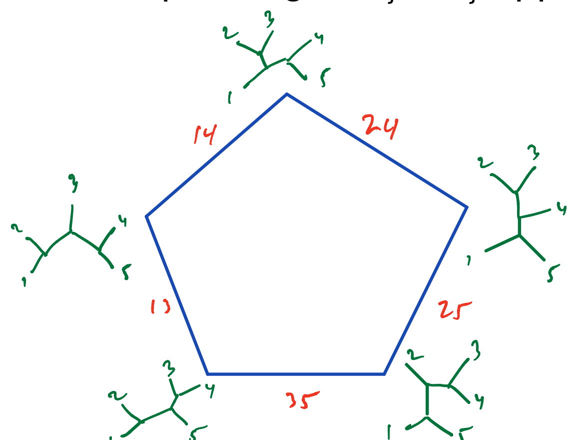
To make the connection to physics, one chooses two orderings of $1, \ldots, n$; we will simplify matters by assuming the orderings are the same. We will consider the following quantity:

$$S_n = \sum_{D \in \mathcal{D}_n} \prod_{e_{ij} \in D} \frac{1}{X_{ij}}$$

We want to relate this to the canonical form of a polyhedron. However, there are some issues, notably that this a function, not a differential form!

The simple associahedron $\mathcal{A}_n$ is, by definition, an $(n-3)$-dimensional polytope whose vertices correspond to the planar, trivalent trees in $\mathcal{D}_n$, and whose facets correspond to the e_{ij}. A vertex D lies on the facet corresponding to e_{ij} if e_{ij} appears in D.





Associahedra were first defined by Jim Stasheff in the 1960s for homotopy-theoretic purposes: planar trivalent trees with n vertices also correspond to ways of parenthesizing an $(n-1)$-term non-associative product.

The vertices of $\mathcal{A}_n$ also correspond to the clusters of the cluster algebra of Dynkin type A_{n-3}; this will come up again at the end.

Realizing the associahedron

The first constructions as polytopes are due to Haiman (1984, unpublished) and Lee (1989).

ABHY give a specific realization of the associahedron, which turns out to have very interesting properties, but they aren't actually essential for what I am attempting here, so I am going to skip the details, and give only the form, which is important.

We consider a space $\mathbb{R}^{n(n-3)/2}$, coordinatized by X_{ij} (of which there are $n(n-3)/2$).

Consider some $(n-3)$-dimensional affine subspace H_n of $\mathbb{R}^{n(n-3)/2}$. Define a polytope

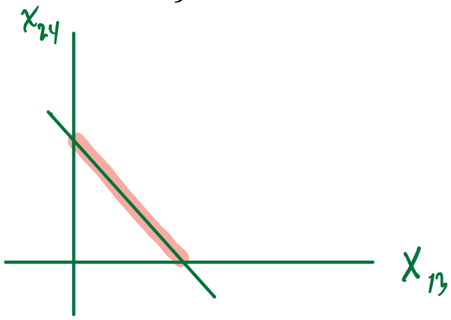
$$Q_n = H_n \cap \mathbb{R}^{n(n-3)/2}_{\geq 0}.$$

Assuming there is some point in H_n which lies in the positive orthant, Q_n is an $(n-3)$-dimensional polytope, cut out by $n(n-3)/2$ inequalities ($X_{ij} \geq 0$).

Any $(n-3)$-dimensional polytope with $n(n-3)/2$ facets can be described in this way.

Realizing the associahedron II

Let's see some examples. The one example which can really be drawn is the $n = 4$ case. We have two X_{ij}, namely X_{13} and X_{24}, and the associahedron is given as $\mathbb{R}^2_{\geq 0} \cap \{X_{13} + X_{24} = c\}$.



For $n = 5$, the description is as the intersection with $\mathbb{R}^5_{\geq 0}$ with a 2-dimensional subspace, cut out by three equations.

ABHY gives a particular realization of $\mathcal{A}_n$ of this type, by defining an $(n-3)$-dimensional subspace H_n and setting

$$\mathcal{A}_n = \mathbb{R}^{n(n-3)/2}_{\geq 0} \cap H_n$$

Most of what I say would work with any associahedron.

Putting the pieces together

We are going to consider the form on $\mathbb{R}^{n(n-3)/2}$ parameterized by X_{ij} (of which there are $n(n-3)/2$).

$$\Omega_n = \sum_{D \in \mathcal{D}_n} \operatorname{sign}(D) \bigwedge_{e_{ij} \in D} \frac{dX_{ij}}{X_{ij}}$$

$\operatorname{sign}(D) \in \{1, -1\}$, and depends on the ordering of the edges in D, and I am not going to be precise about it.

Theorem (Arkani-Hamed, Bai, He, Yan)

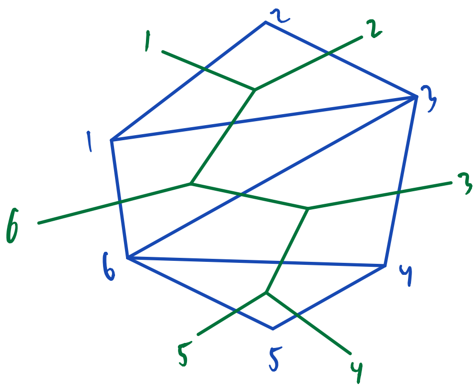
- *The pullback of Ω_n to H_n is the canonical form $\Omega(\mathcal{A}_n)$.*
- *The canonical form of $\mathcal{A}_n$ is the standard volume form times the Feynman diagram sum S_n.*

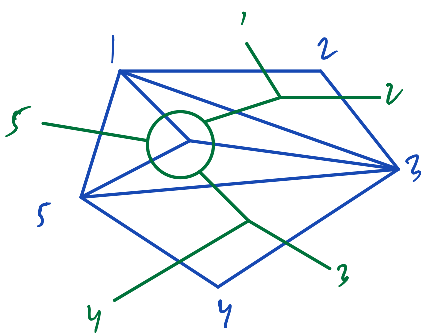
The first statement follows from the description of the canonical form as a sum over the vertices; the second follows by comparing the Feynman diagram sum to the definition of Ω_n, together with a fact about the ABHY realization, which means that $\bigwedge dX_{ij}$ pulls back to $\pm$ the volume form (and some thought about the signs).

Loop levels

The tree-level Feynman diagrams that we have been considering are the first approximation to the amplitude; one needs to allow Feynman diagrams with loops. How should this be thought about?

Look at the tree picture, and take the dual graphs. These give us triangulations of an n-gon, which correspond to the clusters of a Dynkin type A_{n-3} cluster algebra.





The dual of diagrams with one loop correspond (with a caveat) to clusters of a type D_n cluster algebra, triangulations of an n-gon with one puncture.

Loop levels II

Cluster algebraists have shown that D_n cluster algebras have (generalized) associahedra. I showed (with Bazier-Matte, Chapelier, Douville, Mousavand, and Yıldırım) that the ABHY associahedron construction can actually be generalized in a very direct way.

For more at one loop, see work of Giulio Salvatori arXiv:1806.01842 and Arkani-Hamed, He, Salvatori, Thomas arXiv:1912.12948.

There is a cluster algebra associated to a disk with any number of punctures, and it is plausible that these should be related to higher loop corrections.

However, there are some issues that arise. With more than one puncture, the cluster algebra has infinitely many clusters, even though there are only finitely many diagrams. (In fact, there is already a certain mismatch at one loop, but it is relatively benign.) This is a topic of ongoing work.

Thank you!