Positive Geometries

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Definition

$(X, X_{\geq 0})$

positive geometry

X : projective, normal, d-dim variety

$X_{\geq 0}$: closed semialgebraic subset of $X(\mathbb{R})$

$X_{>0} = \mathrm{Int}(X_{\geq 0})$ open oriented d-dim manifold.

$\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$

$\partial X := \overline{\partial X_{\geq 0}} \subset X$ Zariski closure

$= C_1 \cup C_2 \cup \ldots \cup C_r \, (\cup \ldots)$

$d-1$ dim components

$C_{i,\geq 0} = C_i \cap \partial X_{\geq 0}$

$(C_i, C_{i,\geq 0})$ boundary components

A d-dim positive geometry is

- $d=0$ $\quad X_{\geq 0} = X = pt \qquad \Omega(X, X_{\geq 0}) = \pm 1$
- $d>0$ (P1) Every boundary component $(C, C_{\geq 0})$ is a $(d-1)$-dim pos. geom.

(P2) There exists a unique nonzero rational (meromorphic) d-form $\Omega(X, X_{\geq 0})$

canonical form

s.t. $\quad \mathrm{Res}_C \, \Omega(X, X_{\geq 0}) = \Omega(C, C_{\geq 0}) \neq 0$

$\forall (C, C_{\geq 0})$

and no other singularities.

$$\Omega(X, X_{\geq 0}) = \frac{df}{f} \wedge \Omega(C, C_{\geq 0}) + \ldots$$

Examples

$d=1$ $\quad X=$ genus g curve

$\rightsquigarrow$ $g=0$ $\quad X=\mathbb{P}^1$

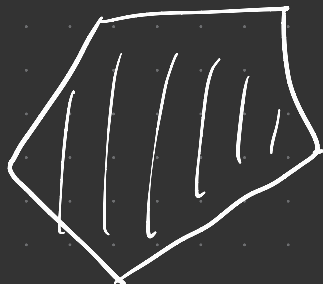
$X_{\geq 0}$ = union of closed intervals

$$\Omega(\mathbb{P}^1,[a,b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{(b-a)}{(b-x)(x-a)}\,dx$$

$$\Omega\Big(\mathbb{P}^1, \coprod_i [a_i,b_i]\Big) = \sum_i \Omega(\mathbb{P}^1,[a_i,b_i])$$

$d=2$ $X = \mathbb{P}^2$ $X_{\geq 0} \subset \mathbb{P}^2(\mathbb{R})$

YES



$$\Omega = C \frac{(1+2y)\,dx\,dy}{(1-x^2-y^2)(\sqrt{3}y+x)(\sqrt{3}y-x)}$$

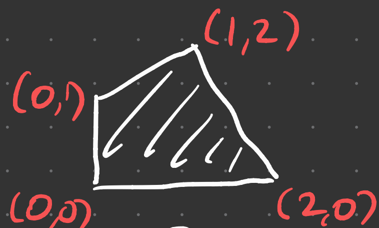
NO

$\Omega = 0$

boundary is elliptic curve

Theorem $P \subset \mathbb{P}^d$ $\dim P = d$ projective polytope is a positive geometry

$K_{\mathbb{P}^2} = \mathcal{O}(-3)$



P

$$\Omega = \frac{C\,(y-4x-4)}{xy\,(y-x-1)(2x+y-4)}\,dx\,dy$$

Idea of proof. $P = \coprod_i T_i$ triangulation

$$\Omega_P = \sum_i \Omega_{T_i}$$

Theorem $P \subset \mathbb{R}^d \subset \mathbb{P}^d$

$$\Omega_P(x) = \mathrm{Vol}\left(\left(P - x\right)^\vee\right)\, dx \qquad \text{for} \quad x \in \mathrm{Int}(P)$$

$$= \frac{\mathrm{adj}_{P^\vee}(x)}{\prod_{\text{facets } F} L_F}\, dx$$

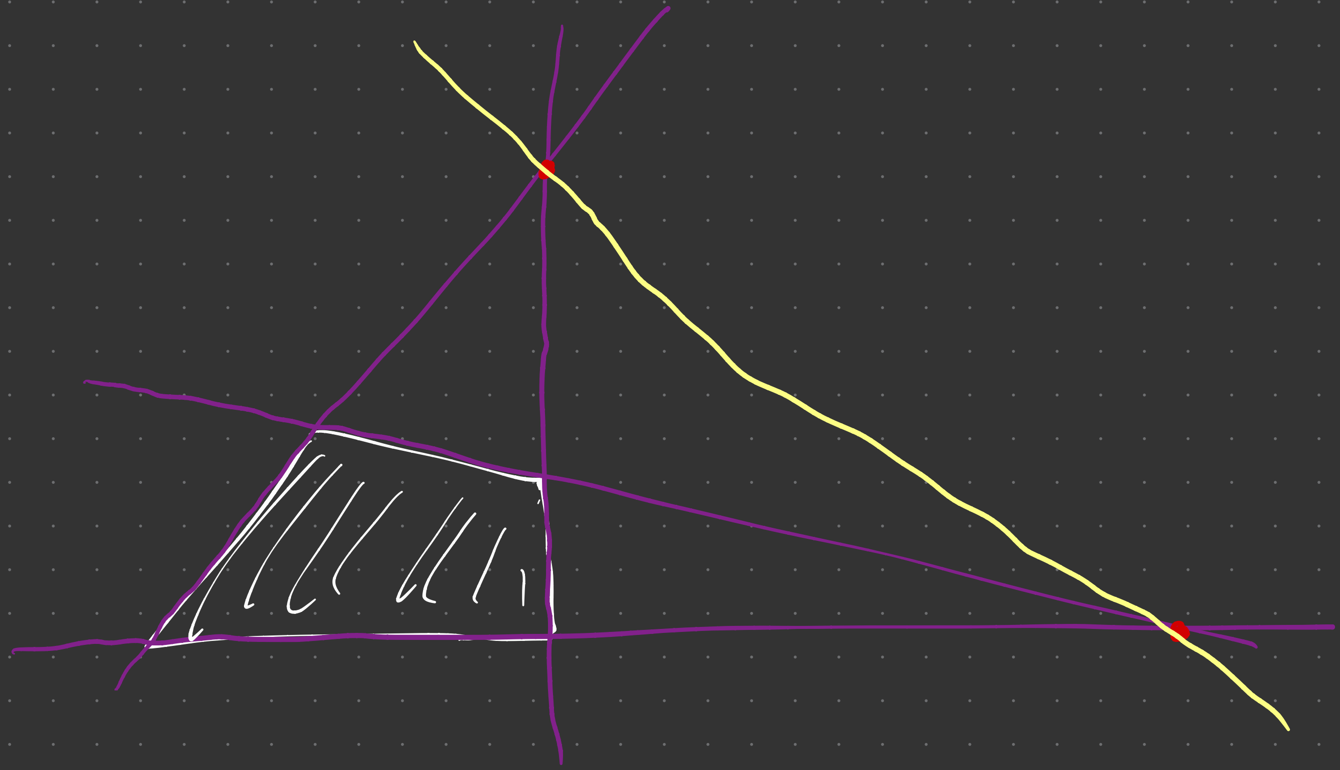
adjoint of $P^\vee$

vanishes on F

Adjoint of $P^\vee$ = polynomial that vanishes on residual arrangement of P [Kohn–Ranestad, Warren]

$\shortparallel$

union of intersections of facet hyperplanes that do not contain any face of P.



$adj_{\Upsilon_{pv}} =$

$(X_P, (X_P)_{\geq 0}) \longrightarrow (\mathbb{P}^d, P)$

Examples

- $(X_P, (X_P)_{\geq 0})$ projective normal toric variety

  $(\mathbb{P}^d, P)$

  $$\Omega = \frac{dx_1}{x_1} \cdots \frac{dx_d}{x_d}$$

- $(Gr(k,n), Gr(k,n)_{\geq 0})$ totally nonnegative Grassmannian
- $(\overline{\mathcal{M}_{0,n}}, (\mathcal{M}_{0,n})_{\geq 0})$ moduli space of n-pointed rational curves
- Conjecture

  $Z : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+m}$

  $(Gr(k,k+m), Z(Gr(k,n)_{\geq 0}))$ Grassmann polytope

  $(Gr(k,k+m), \mathcal{A}_{n,k,m})$ Amplituhedron

General problem:

$$\boxed{\text{Find formulae for } \Omega(X, X_{\geq 0})}$$

- Topology? Contractible, balls, ?
- Face structure?
- Triangulations?
- Convexity? $\Omega_P(x) > 0$ for $x \in \mathrm{Int}(P)$ polytope
  conjectured for $\mathcal{A}_{n,k,m}$ m even
- Pushforward?

Integral functions

$$\int_{X_{>0}} Q\,\Omega(X, X_{\geq 0})$$

regulator

- $$\mathcal{M}(z,\eta) = \int \Omega(\mathcal{A}_{n,k,4}(Z))\,\delta^{4k}(Y;Y_0)\; d^{4N}\phi$$

  N=4 SYM planar tree amplitude

- $$I(S) = \int_{(\mathcal{M}_{0,n})_{>0}} \Omega((\mathcal{M}_{0,n})_{>0}) \left[\begin{matrix}\text{rational}\\ \text{factor}\end{matrix}\right]^S$$

  string amplitude

  $$\int_0^1 u^s(1-u)^t \frac{du}{u(1-u)}$$

  Beta function

- $$\Psi(q) = \int_{Gr(k,n)_{>0}} \Omega(Gr(k,n)_{>0})\; e^{\text{superpotential}}$$

  Whittaker function

  $$\int_0^\infty e^{-(x+\frac{q}{x})} \frac{dx}{x}$$

  Bessel function

  $Gr(1,2)_{>0}$

Stringy canonical forms (Arkani-Hamed, He, L.)

$$I = (\alpha')^d \int_{\mathbb{R}_{>0}^d} \prod_i \frac{dx_i}{x_i} \prod_i x_i^{\alpha' X_i} \prod_{j=1}^{r} P_j(x)^{-\alpha' c_j}$$

$$= I(X_i, c_j, \alpha')$$

where $\alpha' > 0$ string length

$P_j(x)$ nonnegative Laurent polynomials

X_i, c_j parameters $\mathrm{Re}(X_i) > 0$

$\mathrm{Re}(c_j) > 0$

$$I = (\alpha')^d \int \prod_{i=1}^{d} \frac{dx_i}{x_i} \prod_{i=1}^{d} x_i^{\alpha' X_i} \prod_{j=1}^{r} P_j(x)^{-\alpha' c_j}$$

$\mathcal{P}_j :=$ Newton polytope $(P_j(x))$

<u>Theorem</u> [Arkani-Hamed, He, L.]

(1) I converges $\Longleftrightarrow$ $0 \in \mathcal{P} := \sum_{j=1}^{r} c_j \mathcal{P}_j - X$

cf. Berkesch-Forsgard-Passare

(2) $\left(\lim\limits_{\alpha' \to 0} I\right) dX = \mathrm{Vol}(\mathcal{P}^\vee)\, dx = \Omega(\mathcal{P})$

$I(\alpha')$ is a "stringy" deformation of $\Omega(\mathcal{P})$

tree level open string amplitude

$$I_n(s,\alpha') := (\alpha')^{n-3} \int_{(\mathcal{M}_{0,n})_{>0}} \prod_{i<j} (z_i - z_j)^{\alpha' s_{ij}} \; \Omega((\mathcal{M}_{0,n})_{>0})$$

can be written as a stringy canonical form.