– Discussion of Hugh Thomas' talk –

A probabilistic perspective on ϕ^3 theory

► Tom KLOSE

Nikolas TAPIA

Amplituhedron Day

8 April 2021

- **Stochastic Quantisation** goes back to [Parisi & Wu, 1981]:

  consider Euclidean QFT measure as stationary measure of a stochastic process

  - static picture: $\nu(d\phi) := \exp(-S(\phi)) \frac{d\phi}{Z}$ (formal!)

  - Langevin dynamics: $\partial_t \phi = -\frac{\delta S(\phi)}{\delta \phi} + \xi$

    functional derivative

    $\phi = \phi(t,x)$, $t \triangleq$ fictitious time, $x \triangleq$ space-time

    space-time white noise $\mathbb{E}[\xi(t,x)\,\xi(s,y)] = \delta(t-s)\,\delta^{(d)}(x-y)$

- $S(\phi) := \int_{\mathbb{R}^4} \frac{1}{2} |\nabla \phi(x)|^2 + \frac{1}{3} \phi(x)^3 \, d^{(d)}x$

  $\longrightarrow$ $\boxed{\partial_t \phi = \Delta \phi - \phi^2 + \xi}$ (ϕ^3 equation on $\mathbb{R}_+ \times \mathbb{T}^d$, $\phi(0,\cdot) = 0$)

- Fixed-point argument for $\boxed{}$: $\quad \phi = P * (-\phi^2 + \xi)$

  - $\phi^{(0)} = 0 \longrightarrow \phi^{(1)} = P * \xi =: \overset{\circ}{|}$

    $\boxed{\text{NB:}}$ $\quad \xi \in C^{-\frac{d+2}{2}-\kappa}$ a.s. $\Longrightarrow \overset{\circ}{|} \in C^{\overset{-1-\kappa}{-3+2-\kappa}}$ a.s. (Schauder)

  - $\phi^{(2)} = P * (-\boxed{\overset{\circ}{|}^{\,2}} + \xi)$

    ill-posed : $C^\alpha \times C^\beta \ni (f,g) \longmapsto f \cdot g \in C^{\alpha \wedge \beta}$ is well-def IFF $\alpha + \beta > 0$.

- Cure: consider abstract FP problem [Hairer]:

$$\boxed{\Phi = P\left(-\Phi^2 + \circ\right)}$$

$\xrightarrow{\text{Iterate!}}$ $$\Phi(z) = \underset{-1-\kappa}{\overset{\circ}{|}} - \underset{-2\kappa}{\overset{\circ\;\circ}{\mathsf{Y}}} + \varphi(z)\,\underset{0}{\mathbf{1}} - 2\,\underset{1-3\kappa}{\mathsf{Y}^{\circ\circ\circ}} - 2\varphi(z)\,\underset{1-\kappa}{\big\langle^{\circ}} + \langle \nabla\varphi(z), \underset{1}{X} \rangle$$

- $\Pi^{\varepsilon} \tau \in \mathcal{S}'(\mathbb{R}^{4+1})$ w/
  - $\Pi^{\varepsilon} \circ := \xi_{\varepsilon} \triangleq$ smoothened STWN
  - $\Pi^{\varepsilon} \overset{\tau}{\mid} := P * \Pi^{\varepsilon} \tau$
  - $\Pi^{\varepsilon} \tau \bar{\tau} := \Pi^{\varepsilon} \tau \; \Pi^{\varepsilon} \bar{\tau}$

<u>Ex.:</u> $\Pi^{\varepsilon} \mathrm{Y} = P * \Pi^{\varepsilon} \vee = P * (\Pi^{\varepsilon} \mid)^2 = P * \mid^{2}_{\varepsilon} \triangleq$ regularised at scale ε

- $\boxed{\text{Problem in regularity structures:}}$

counterterms to ensure ex. of $\hat{\phi} := \lim_{\varepsilon \downarrow 0} \hat{\phi}_{\varepsilon}$.

$$(\partial_t - \Delta)\hat{\phi}_{\varepsilon} = -\hat{\phi}_{\varepsilon}^{2} + \xi_{\varepsilon} \boxed{+ \sum_{\deg(\tau)<0} C_{\tau}(\varepsilon)\, \Upsilon_{\tau}^{RHS}(\hat{\phi}_{\varepsilon})}$$

[Bruned, Chandra, Chevyrev, Hairer]

with $C_{\tau}(\varepsilon) := \mathbb{E}[\Pi^{\varepsilon} \mathcal{A}_{-} \tau(0)]$

$\boxed{\text{Negative twisted antipode: "Extraction - contraction procedure"}}$

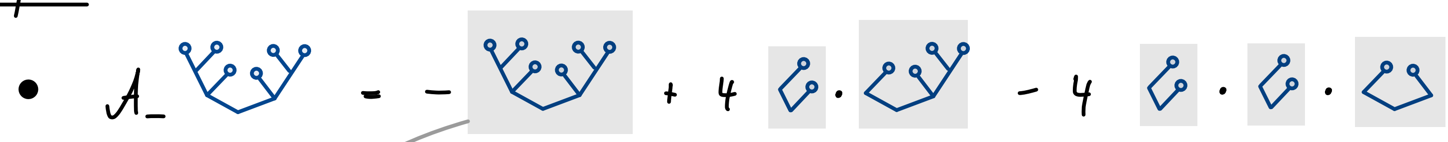
↗ [Bruned, Hairer, Zambotti] & [Chandra, Hairer]

- Explicitly for the ϕ^3 eq. [Beglund & Bruned]:

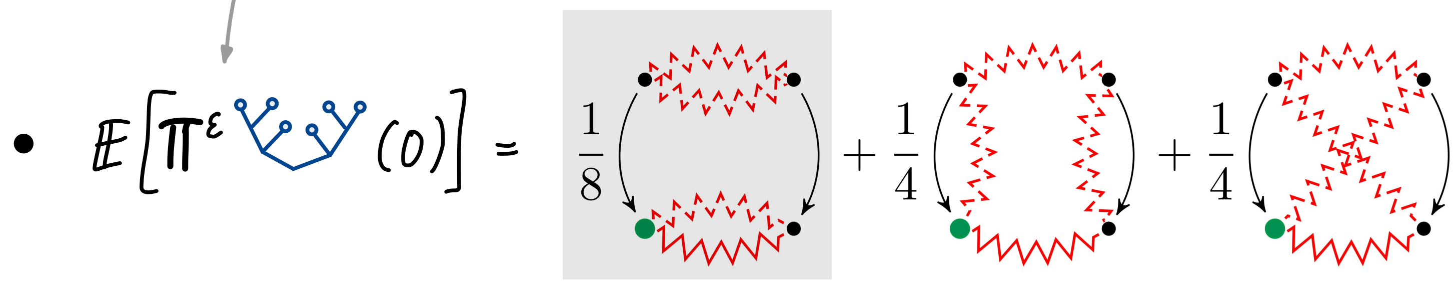
$$\partial_t \hat{\phi}_\varepsilon = \Delta \hat{\phi}_\varepsilon - \hat{\phi}_\varepsilon^2 + \xi_\varepsilon + C_0(\varepsilon) + C_1(\varepsilon)\, \hat{\phi}_\varepsilon$$

▸ $C_i(\varepsilon) = \mathbb{E}[\,\Pi^\varepsilon A_- \tau_i\,(0)]$ for some τ_i w/ $\deg(\tau_i) < 0$

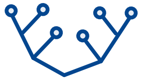
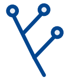
<u>Example:</u>



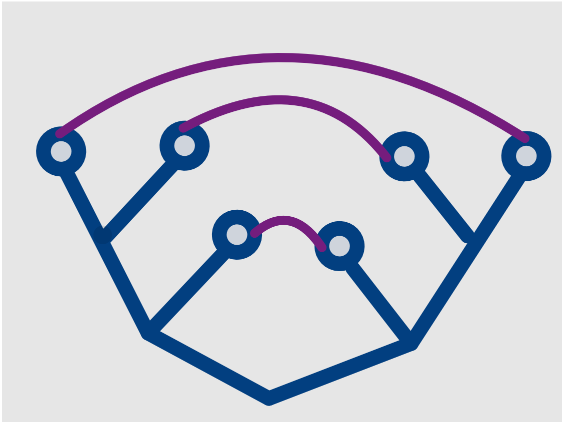
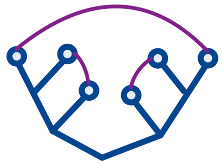
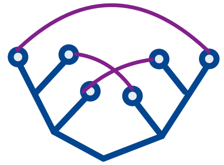
▸ Each of the shaded trees gives a (sum of) vacuum Feynman diagram!
(Π^ε acts multiplicatively on forests)



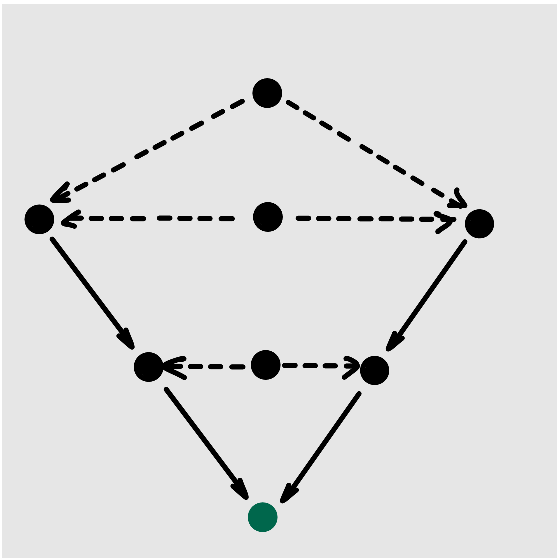
Let's look at this one in some more detail!

$$\mathbb{E}\Big[\Pi^{\varepsilon}$$  $$(0)\Big] = \mathbb{E}\Big[\big(\Pi^{\varepsilon}$$  $$(0)\big)^2\Big] = \mathbb{E}\Big[\int [\dots]\, \xi^{\otimes 6}(dz_1 \cdot \ldots \cdot dz_6)\Big]$$

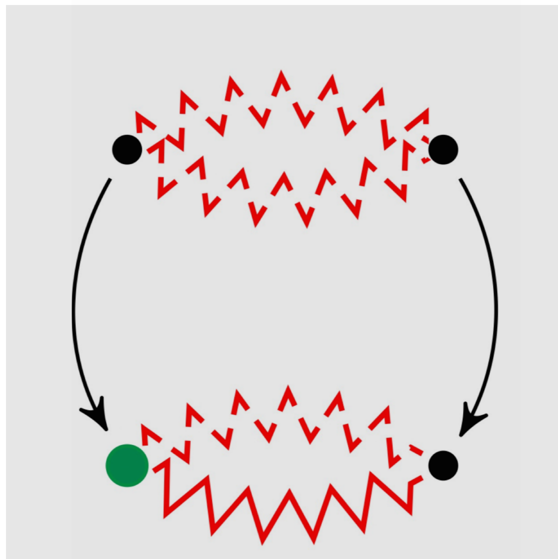
+ use Wick's thm!

= 2  + 4  + 4  + [0's]

convert

2 

conversion rule

$\frac{1}{8}$ 

- These computations become unwieldy very quickly!

$\longrightarrow$ We need to exploit cancellations for $\hat{\phi} := \lim_{\varepsilon \downarrow 0} \hat{\phi}_\varepsilon$ to exist!

$\boxed{?}$ In the spirit of the original motivation:

Can we use AMPLITUHEDRON-like structures to organise our Feynman diagram computations?

(RS allow to treat locally subcritical $\equiv$ superrenormalisable stochastic PDEs,

i.e. $\# [\tau : \deg(\tau) < 0] < \infty$.

Even more important when one wants to study critical SPDEs!)

THANK YOU!