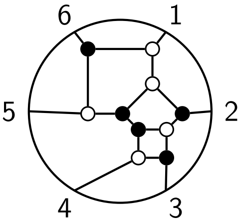
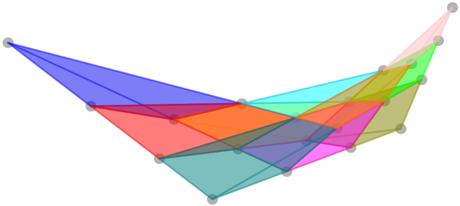
Introduction to the amplituhedron

Slides available at `http://lacim-membre.uqam.ca/~karp`





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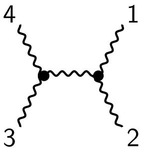
Amplituhedron Day

Scattering amplitudes

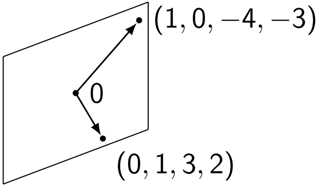
- A *scattering amplitude* is a function associated to a process of interacting particles.
- We will work in *planar* $\mathcal{N} = 4$ *supersymmetric Yang–Mills theory*, and fix two parameters n and k, where n is the number of particles, $k+2$ of which have helicity $-$ and $n-k-2$ of which have helicity $+$.
- Classically, scattering amplitudes are calculated as a sum over Feynman diagrams:



- For example, for $n = 6$ and $k = 0$, there are 220 Feynman diagrams. Yet, Parke and Taylor (1986) discovered that the scattering amplitude can be expressed as a single term.
- For general n and k, the scattering amplitude is encoded in a geometric object called the *amplituhedron*.

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$C :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

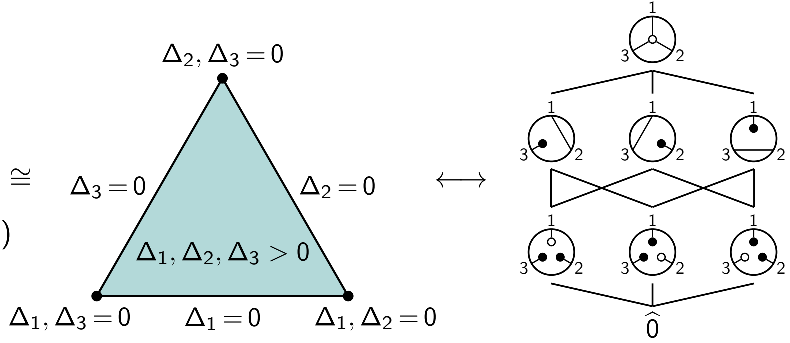
$$\Delta_{12} = 1, \;\; \Delta_{13} = 3, \;\; \Delta_{14} = 2, \;\; \Delta_{23} = 4, \;\; \Delta_{24} = 3, \;\; \Delta_{34} = 1$$

- Given $C \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(C)$ be the $k \times k$ minor of C in columns I. The *Plücker coordinates* $\Delta_I(C)$ are well defined up to a common nonzero scalar.
- We call $C \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(C) \geq 0$ for all k-subsets I. The set of all such C forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- When $k = 1$, the Grassmannian $\mathrm{Gr}_{1,n}$ specializes to projective space $\mathbb{P}^{n-1}$, the set of nonzero vectors $(x_1 : \cdots : x_n)$ modulo rescaling.

The positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (`alg-geom/9709035`) and Postnikov (`math/0609764`). Each *positroid cell* is specified by requiring some Plücker coordinates to be strictly positive, and the rest to be zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$
$(= \mathbb{P}_{\geq 0}^{2})$



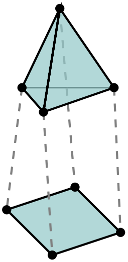
- $\mathrm{Gr}_{1,n}^{\geq 0}$ is the standard $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$:

$$\{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1} \,:\, x_1, \cdots, x_n \geq 0, \;\; x_1 + \cdots + x_n = 1\}.$$

We can view $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex into the Grassmannian.

Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



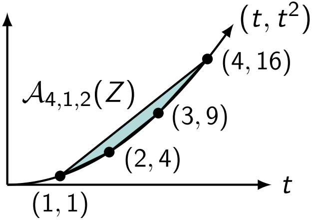
simplex in $\mathbb{P}^{n-1}$	- - - - - - - - →	$C \in \mathrm{Gr}_{k,n}^{\geq 0}$
↓ linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$		↓ linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$
polytope in $\mathbb{P}^m$	- - - - - - - - →	*Grassmann polytope* $CZ^T \in \mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called the *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$. Amplituhedra were introduced by Arkani-Hamed and Trnka (`1312.2007`), and inspired Lam (`1506.00603`) to define Grassmann polytopes. The case relevant to physics is $m = 4$, but $\mathcal{A}_{n,k,m}(Z)$ is an interesting space for any m.

$k = 1$: cyclic polytopes

- One way to construct Z with positive maximal minors is to take n points on the *moment curve* $(t, t^2, \cdots, t^{k+m-1})$ in $\mathbb{R}^{k+m-1}$.
- e.g. $n = 4$, $k + m = 3$



$$\longleftrightarrow \qquad Z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

- When $k = 1$, the amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is the polytope in $\mathbb{P}^m$ whose vertices are the columns of Z.
- e.g.

$$(x_1 : x_2 : x_3 : x_4) \quad \mapsto \quad x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}$$

$$\in \mathbb{P}^3_{\geq 0} = \mathrm{Gr}^{\geq 0}_{k,n} \qquad\qquad \in \mathcal{A}_{4,1,2}(Z) \subseteq \mathbb{P}^2 = \mathrm{Gr}_{k,k+m}$$

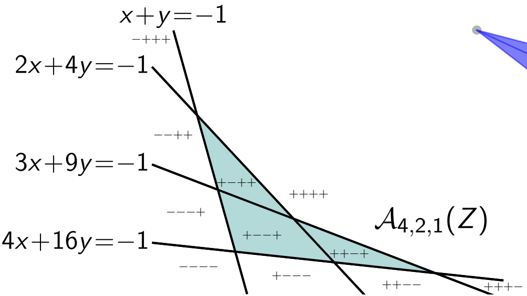
- Sturmfels (1988): every amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope*.

$m = 1$: cyclic hyperplane arrangements

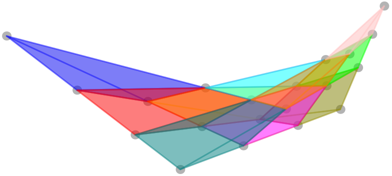
- A *cyclic hyperplane arrangement* consists of n hyperplanes of the form

$$tx_1 + t^2x_2 + \cdots + t^kx_k + 1 = 0 \text{ in } \mathbb{R}^k \quad (t > 0).$$

- e.g. $n = 4$, $k = 2$ $\qquad$ $n = 6$, $k = 3$



$\mathcal{A}_{4,2,1}(Z)$

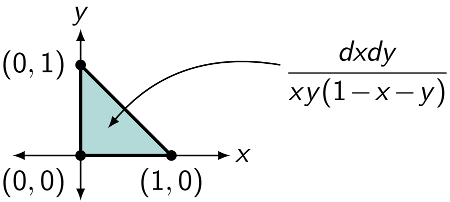


$\mathcal{A}_{6,3,1}(Z)$

- Karp, Williams (1608.08288): $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$.
- Karp, Williams; Arkani-Hamed, Thomas, Trnka (1704.05069): conjectural characterizations of $\mathcal{A}_{n,k,m}(Z)$ in terms of *sign variation*.

Positive geometries and differential forms

- Arkani-Hamed, Bai, Lam (1703.04541): a *positive geometry* is a space equipped with a differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



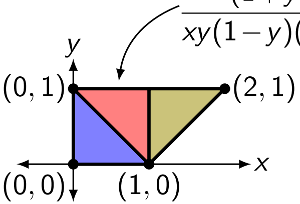
- $\mathrm{Gr}_{k,n}^{\geq 0}$ is a positive geometry. The differential form for e.g. $\mathrm{Gr}_{2,4}^{\geq 0}$ is

$$\frac{dxdydzdw}{\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}}, \text{ where } C = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix} \in \mathrm{Gr}_{2,4}\,.$$

- The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is conjecturally a positive geometry, whose differential form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.

Triangulations and duality

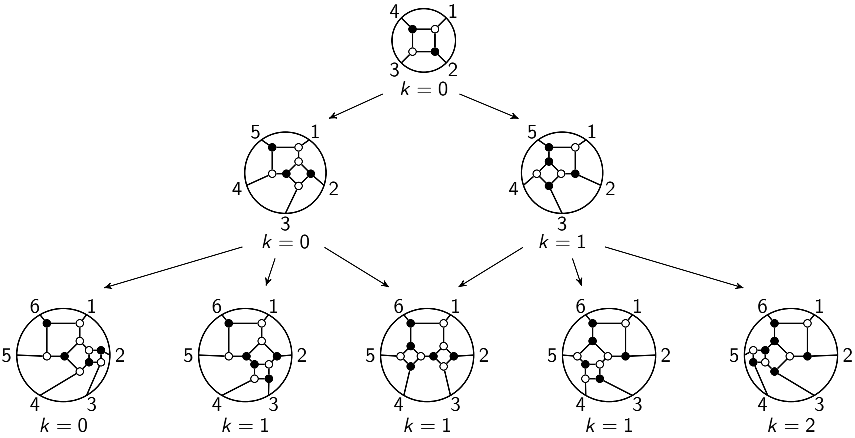
- *Triangulation* is one way to obtain the differential form of a positive geometry:



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

- Arkani-Hamed, Trnka: the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is conjecturally triangulated by the images under Z of certain $4k$-dimensional positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. These cells come from the *BCFW recursion* (`hep-th/0412308`, `hep-th/0501052`) for the scattering amplitude.
- The differential form of any polytope can be expressed as the volume of its dual (polar) polytope. Can we find a triangulation-independent formula for the amplituhedron form? Is it the volume of a *dual amplituhedron*?

$m = 4$: BCFW recursion

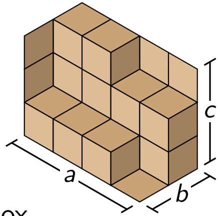


- The conjectured BCFW triangulation of $\mathcal{A}_{n,k,4}(Z)$ uses $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$ cells. This is a *Narayana number*, a refinement of the *Catalan number*.
- Karp, Williams, Zhang, and Thomas (1708.09525): interpretations of the cells in terms of binary trees, pairs of lattice paths, and Dyck paths.

General m even: plane partitions?

- Karp, Williams, and Zhang conjecture that for m even, $\mathcal{A}_{n,k,m}(Z)$ has a triangulation into $M(k, n-k-m, \frac{m}{2})$ cells, where

$$M(a,b,c) := \prod_{p=1}^{a} \prod_{q=1}^{b} \prod_{r=1}^{c} \frac{p+q+r-1}{p+q+r-2}$$



is the number of plane partitions inside an $a \times b \times c$ box.

- $M(a,b,c)$ is symmetric in (a,b,c). The $k \leftrightarrow n-k-m$ symmetry was explained by Galashin and Lam (1805.00600) using the *twist map*. When $m = 4$, this comes from *parity* (symmetry of the helicities $+$ and $-$) of the scattering amplitude. The possible $k \leftrightarrow \frac{m}{2}$ symmetry is mysterious.
- Mohammadi, Monin, and Parisi (2010.07254) defined the *secondary amplituhedron* of $\mathcal{A}_{n,k,m}(Z)$ when $n-k-m = 1$.
- For any m, it is expected that $\mathcal{A}_{n,k,m}(Z)$ is a *regular CW complex* homeomorphic to a closed ball. This is known only in special cases.

Beyond amplituhedra

- Loop amplituhedra (1312.2007): positive geometries for scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, for any loop order $L \geq 0$. (When $L = 0$, we get $\mathcal{A}_{n,k,m}(Z)$.)
- Cosmological polytopes (1709.02813): positive geometries for the wavefunction of the universe in certain toy models.
- Associahedra (1711.09102): positive geometries for tree-level scattering amplitudes in bi-adjoint ϕ^3 scalar theory.
- Stokes polytopes and accordiohedra (1906.02985): positive geometries for tree-level scattering amplitudes in ϕ^p theory.
- Momentum amplituhedra (1905.04216): positive geometries for tree-level $\mathcal{N} = 4$ scattering amplitudes in spinor helicity space.
- EFThedra (2012.15849): spaces exhibiting causality and unitarity constraints for 4-particle scattering amplitudes in effective field theories.

Thank you!